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Scientific Report 1

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ABSTRACT

The propagation of waves in both fully and partially ionized gases, both with and without magnetic fields, has been treated by a number of workers; e. g. Tanenbaum and Mintzer in 1962 obtained the dispersion relations for a linearized and spatially uniform gas of electrons, positive ions and neutrals. The present report discusses the basic formulation and mathematical treatment of wave propagation in a linearized electron-ion-neutral gas, with static magnetic field, in which the ambient gas parameters have an arbitrary variation with the vertical coordinate and are uniform in the horizontal direction.

The first part of the report discusses a rather standard formulation of the general problem, via the Boltzmann equation and the Maxwell equations. By appropriate momentum-space averaging, the Boltzmann equation yields motion, continuity and dynamic adiabatic state equations. These are then combined to yield neutral and composite plasma equations of motion, continuity and adiabatic state and a generalized Ohm's Law. Steady state plane-wave solutions are suitable in the horizontal coordinates, reducing x, y and t dependence to algebra but the equations must remain differential in the vertical coordinate z. This gives rise to a system of 10 simultaneous, highly coupled ordinary first order differential equations and 11 simultaneous algebraic equations.

The second part of the report is a discussion of the mathematical solutions of this coupled algebraic differential equation system, which is equivalent to the system of equations arising in analysis of coupled linear electrical networks. Referring to both the mathematical literature of differential equations and the modern "state-space" approach to automatic control systems, various

purely analytical approaches are discussed with emphasis on their deficiencies in obtaining practical numerical results with an arbitrary z-variation. The Runge-Kutta step-by-step procedure was eventually invoked and a Fortran program was written based on this technique. The program can be used to obtain accurate numerical solutions to many problems involving wave propagation in a linearized, vertically non-uniform electron-ion-neutral gas without the necessity for making drastic simplifying assumptions for the vertical non-uniformity. This program can be used to treat, by changing input parameter values, such diverse problems as the perturbing effect of acoustic-gravity waves on ionospheric electron density, electromagnetic wave propagation in the vertically inhomogeneous ionosphere, MHD waves high in the ionosphere, or other kinds of wave propagation in plasma media with a vertical inhomogeneity. Numerical solutions for the acoustic-gravity wave-plasma interaction problem and their interpretation will be presented in a later report.

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1. Introduction

Dispersion relations for a homogeneous, linearized three fluid gas, consisting of electrons, ions and neutrals, with a static magnetic field, have been investigated by a number of workers, including Tanenbaum and Mintzer[1] and more recently by Cronson and Clark[2]. The latter focusses attention on the specific problem of the ionosphere. A very thorough analytical and numerical study of the dispersion relations at frequencies below a few cycles per second, using the latest and most reliable ionospheric data at the time of writing (June, 1964), shows that of the four permissible modes of propagation, (neutral acoustic, plasma acoustic and two Alfven modes), the only ones that are not prohibitively damped out by collisions between charged and neutral particles are neutral acoustic waves. Perturbation of the plasma gas at appreciable distances from the exciting source at low frequencies are very likely to be due to such waves. The extreme damping of other plasma-induced propagation modes is due to the very small fractional ionization in the ionosphere.

Vertical gradients in static ionospheric parameters, e.g. neutral gas density, plasma density and collision frequency are in some cases sufficiently steep so that the parameters change appreciably within a wavelength of these low frequency acoustic waves. For this reason, an analysis assuming complete homogeneity of the medium might exclude some important effects. In particular, Hines[3,4,5] and Pitteway[4] have analyzed the propagation of gravity waves that arise in a neutral gas with a vertical gradient in ambient gas pressure and

density, both with[4,5] and without[3] consideration of viscosity and heat conduction. An analysis not accounting explicitly for these gradients would not have predicted the existence of such waves.

The present work could be considered as a generalization of Hines 1960 paper[3] to include the interaction of the gravity waves propagating in the neutral gas with the electron-ion plasma embedded in the gas. From another point of view, it could be considered as a generalization of the work of Tanenbaum and Mintzer[1] or Cronson and Clark[2] to include the effect of vertical nonuniformity on wave propagation in a composite gas consisting of electrons, ions and neutral particles.

The present paper covers the formulation of the general problem and its mathematical solution. The investigation of the analytical problem that finally led to the use of a numerical technique for solution is described in some detail. The numerical results and their interpretation will be treated in a later report.

2. Formulation of the General Problem

The formulation of the general problem (under the assumption that no ionization processes take place on the time scale of the phenomena of interest) begins with the Boltzmann equation for each of M constituent ideal gases and the Maxwell equations for electric and magnetic fields. These are:

$$\frac{dn_k}{dt} = \frac{\partial n_k}{\partial t} + \sum_{i=1}^{3} \frac{\partial n_k}{\partial x^{(i)}} u^{(i)} + \sum_{i=1}^{3} \frac{\partial n_k}{\partial u^{(i)}} \frac{F_k^{(i)}}{m^{(k)}} = \left(\frac{\partial n_k}{\partial t}\right)_c ; \; k = 1, \ldots M \tag{1}$$

$$\nabla \times \underset{\rightarrow}{e} = - \mu_o \frac{\partial \underset{\rightarrow}{h}}{\partial t} \tag{2}$$

$$\nabla \times \underset{\rightarrow}{h} = \underset{\rightarrow}{j} + \epsilon_o \frac{\partial \underset{\rightarrow}{e}}{\partial t} \tag{3}$$

where

$n_k \equiv n_k(\underset{\rightarrow}{r}, \underset{\rightarrow}{u}; t)$ = number density of particles of the k^{th} species at a point $(\underset{\rightarrow}{r}, \underset{\rightarrow}{u})$ in phase-space at time t,

$\left(\frac{\partial n_k}{\partial t}\right)_c$ = partial time-rate of change of n_k due to collision effects,

where $\underset{\rightarrow}{r} = \left(x^{(1)}, x^{(2)}, x^{(3)}\right)$ represents position, and

$\underset{\rightarrow}{u} = \left(u^{(1)}, u^{(2)}, u^{(3)}\right)$ the velocity vector, where "phase space" is defined as the six-dimensional space of position and velocity (rather than position and momentum).

$\underset{\rightarrow}{F}_k = \left(F_k^{(1)}, F_k^{(2)}, F_k^{(3)}\right)$ = force on particles of k^{th} species.

$\underset{\rightarrow}{e} \equiv$ electric field vector in volts/meter

$\underset{\rightarrow}{h} \equiv$ magnetic vector in amperes/meter

$\underset{\rightarrow}{j} \equiv$ current density vector in amperes/meter^2

μ_o = magnetic permeability of free space = $4\pi \times 10^{-7}$ henry/meter

ϵ_o = dielectric coefficient of free space = $\frac{1}{36\pi} \times 10^{-9}$ farad/meter

Note that all quantities are in MKS units.

In the standard way, following Spitzer[6]; we define an average over all velocity space of a general function G(u) as:

$$\overline{G(\underset{\sim}{u})} \equiv \int\!\!\int\!\!\int_{\text{All } \underset{\sim}{u}\text{-space}} G(\underset{\sim}{u}) n_k(\underset{\sim}{r}, \underset{\sim}{u}; t)\, d^3\underset{\sim}{u} \Big/ \int\!\!\int\!\!\int_{\text{All } \underset{\sim}{u}\text{-space}} n_k(\underset{\sim}{r}, \underset{\sim}{u}; t) d^3\underset{\sim}{u} \tag{4}$$

and note that the number density of k-type particles of all velocities at r at time t is

$$N_k(\underset{\sim}{r}, t) \equiv \int\!\!\int\!\!\int_{\text{All u-space}} n_k(\underset{\sim}{r}, \underset{\sim}{u}; t) d^3\underset{\sim}{u} \tag{5}$$

By setting $G(\underset{\sim}{u}) = 1$ in (4), we are led to an equation of continuity or mass or number conservation in each of the constituent gases. By setting $G(\underset{\sim}{u})$ equal to a velocity component, we are led to equations of motion for each gas. Two key assumptions in these developments are:

(a) The i^{th} component of the average force $\overline{\underset{\sim}{F}^{(i)}}$ is independent of (i) but may depend on $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ and/or $u^{(j)}$ where $j \neq i$. Note that electric and gravitational forces are independent of velocity while the i^{th} component of the Lorentz forces $\mu_o(\underset{\sim}{u} \times \underset{\sim}{h})$ depend only on velocity components other than $u^{(i)}$. These are the three types of non-collision forces that will be considered in our analysis. Collision effects are contained in the term $\left(\frac{\partial n_k}{\partial t}\right)_c$ and hence are not included in the force $\underset{\sim}{F}_k$.

(b) The number density $N_k(\underset{\sim}{r}, t)$ at a point $\underset{\sim}{r}$ is not changed by collisions, i.e.

$$\left[\frac{\partial N_k(\underset{\sim}{r}; t)}{\partial t}\right]_c = 0 \tag{6}$$

where the subscript c denotes "due to collisions".

If assumptions (a) and (b) are invoked in (1) and the averaging process indicated by (4) and (5) is carried out with $G(\underset{\sim}{u}) = 1$, the result is a set of continuity or number-conservation equations.

$$\nabla \cdot (N_k \bar{\underset{\rightarrow}{u}}_k) + \frac{\partial N_k}{\partial t} = 0;\ k = 1, \ldots, M \tag{7}$$

where $\bar{\underset{\rightarrow}{u}}_k$ is the average velocity of particles of the $k^{\underline{th}}$ species or equivalently

$$\nabla \cdot (\rho_k \bar{\underset{\rightarrow}{u}}_k) + \frac{\partial \rho_k}{\partial t} = 0;\ k = 1, \ldots, M \tag{7}'$$

where $\rho_k \equiv m_k N_k$, the mass-density of k-type particles.

Application of the same averaging process with the assumptions (a) and (b) to the case where

$$G(\underset{\rightarrow}{u}) = u^{(i)};\ i = 1, 2, 3 \tag{8}$$

results in the vector equation of motion for the $k^{\underline{th}}$ particle species, i.e.

$$\rho_k \frac{D\underset{\rightarrow}{u}_k}{Dt} \equiv \rho_k \left(\frac{\partial \underset{\rightarrow}{u}_k}{\partial t} + \bar{\underset{\rightarrow}{u}}_k \cdot \nabla \bar{\underset{\rightarrow}{u}}_k \right) = - \nabla \cdot \underline{\Psi}^{(k)} + \frac{\rho_k q_k}{mk} (\underset{\rightarrow}{e} + \mu_o \bar{\underset{\rightarrow}{u}}_k \times \underset{\rightarrow}{h})$$

$$+ \rho_k \nabla \phi_g + \sum_{\substack{j=1 \\ j \neq k}}^{M}{}' \rho_k \nu_{kj} (\bar{\underset{\rightarrow}{u}}_k - \bar{\underset{\rightarrow}{u}}_j) \tag{9}$$

where the forces accounted for are electric, magnetic, and gravitational field forces, the latter being proportional to the gradient of a gravitational potential function ϕ_g, q_k is the charge of a particle of the $k^{\underline{th}}$ species; the collision forces are assumed to be of the simple "frictional" type, as obtained from simple kinetic theory arguments, i.e. proportional to the velocity difference between colliding particles. The collision frequencies are given by ν_{kj}, and the stress tensor $\underline{\Psi}^{(k)}$ is defined as a matrix whose elements are:

$$\Psi_{\ell m}^{(k)} \equiv \rho_k \overline{(u_k^{(\ell)} - \bar{u}_k^{(\ell)})(u_k^{(m)} - \bar{u}_k^{(m)})} \quad . \tag{10}$$

Assumption of a Maxwell-Boltzmann distribution of velocities diagonalizes the stress-tensor and by very elementary kinetic theory arguments, we obtain

$$\psi_{\ell m}^{(k)} = \frac{N_k m_k \overline{v_k^2}}{3} = p_k \tag{11}$$

where $\overline{v_k^2} = \sum_{\ell=1}^{3} \overline{\left[u_k^{(\ell)} - \bar{u}_k^{(\ell)}\right]^2}$ = mean square deviation of k-type particle speed

p_k = scalar pressure of k^{th} gas species.

Auxiliary to eq. (9) is the statement that the net momentum exchange in collisions between any two types of particles must be zero, required by Newton's third law. Thus:

$$\rho_k \nu_{kj} (\bar{\underset{\sim}{u}}_k - \bar{\underset{\sim}{u}}_j) = -\rho_j \nu_{jk} (\underset{\sim}{u}_j - \underset{\sim}{u}_k) \; ; \; j, k = 1, \ldots, M \tag{12}$$

as a consequence of which

$$\frac{\nu_{kj}}{\nu_{jk}} = \frac{\rho_j}{\rho_k} = \frac{m_j N_j}{m_k N_k} \tag{13}$$

We now perform the standard linearization procedure on the equations (2), (3) (7) or (7)' and (9); i.e. we assume that the variables N_k or ρ_k, p_k and the components of $\underset{\sim}{h}$ each consist of a large zero-order part, denoted with subscript 0, plus a small first order perturbation part, denoted with subscript 1, while the components of $\underset{\sim}{u}_k$ and $\underset{\sim}{e}$ are assumed to have vanishing first order parts, i.e. to be of "small perturbation" magnitude. We then neglect all terms of second order, e.g. the term $\bar{\underset{\sim}{u}}_k \cdot \nabla \bar{\underset{\sim}{u}}_k$ in (9) or the term $\nabla \cdot (\rho_{k1} \bar{\underset{\sim}{u}}_k)$ in (7)'. We then assume that the pressure and density of each gas are related by an adiabatic state equation whose general and linearized forms are:

$$\frac{d}{dt}\left(\frac{p_k}{\rho_k^{\gamma}}\right) = \frac{\partial}{\partial t}\left(\frac{p_k}{\rho_k^{\gamma}}\right) + \bar{\underset{\sim}{u}}_k \cdot \nabla\left(\frac{p_k}{\rho_k^{\gamma}}\right) = 0 \text{ (general)} \tag{14}$$

$$\frac{\partial p_{k1}}{\partial t} - c_{sko}^{\,2}\frac{\partial \rho_{k1}}{\partial t} = -\bar{\underset{\sim}{u}}_k \cdot [\nabla p_{ko} - c_{sko}^{\,2}\nabla \rho_{ko}] \quad \text{(linearized)} \qquad (14)'$$

where γ is the ratio of specific heat at constant pressure to that at constant volume, assumed to be the same for all gases, and $c_{sko}^{\,2} \equiv \frac{\gamma p_{ko}}{\rho_{ko}}$ is the zero order sound velocity in the $k^{\underline{th}}$ gas species. Eqn. (14) is based on the assumption that no heat flows in or out during the short periods associated with the perturbation of the gas, although the ambient or "zero order" state of the gas is by no means necessarily adiabatic.

Finally, we write the set of linearized equations required to describe the system, dropping the lines over the $\bar{\underset{\sim}{u}}_k$'s which indicate averaging, as follows:

$$\text{I-k; } \rho_{ko}\frac{\partial \underset{\sim}{u}_k}{\partial t} = -\nabla p_{k1} + \frac{\rho_{ko} q_k}{m_k}(\underset{\sim}{e} + \mu_o \underset{\sim}{u}_k \times \underset{\sim}{h}_o) + \rho_{k1}\nabla\phi_g + \sum_{\substack{j=1 \\ j\neq k}}^{M}{}' \rho_{ko}\nu_{kj}(\underset{\sim}{u}_k - \underset{\sim}{u}_j)\ ;\ k = 1, \ldots, M \qquad \text{Eq. (9)}$$

$$\text{II-k; } \nabla \cdot (\rho_{ko}\underset{\sim}{u}_k) + \frac{\partial p_{k1}}{\partial t} = 0\ ;\ k = 1, \ldots, M \qquad \text{Eq. (7)}'$$

$$\text{III-k; } \frac{\partial p_{k1}}{\partial t} - c_{sko}^{\,2}\frac{\partial \rho_{k1}}{\partial t} = -\underset{\sim}{u}_k \cdot [\nabla p_{ko} - c_{sko}^{\,2}\nabla \rho_{ko}];\ k = 1, \ldots, M \qquad \text{Eq. (14)}'$$

$$\text{IV; } \nabla \times \underset{\sim}{e} = -\mu_o \frac{\partial \underset{\sim}{h}_1}{\partial t} \qquad \text{Eq. (2)}$$

$$\text{V; } \nabla \times \underset{\sim}{h}_1 = \underset{\sim}{j} + \epsilon_o \frac{\partial \underset{\sim}{e}}{\partial t} \qquad \text{Eq. (3)}$$

where the current density j is defined by

$$\underset{\sim}{j} = \sum_{k=1}^{M} \underset{\sim}{j}_k \qquad (15)$$

where $\underset{\sim}{j}_k \equiv N_{ko} q_k \underset{\sim}{u}_k = \frac{\rho_{ko} q_k \underset{\sim}{u}_k}{mk}$.

Note that the condition expressed by eq. (13) must hold.

Each of the vector equations I-k, IV and V has 3 component equations (which we will denote by eq. I-k-1, I-k-2, I-k-3, IV-1, ... etc.). Thus the system I-k through V contains a total of (3M + M + M + 3 + 3) = (5M + 6) equations. The zero order parameters are assumed to be known. The unknowns are the perturbed or first order parameters, i.e. all components of $\underset{\sim}{u}_k$'s, constituting a total of 3M, all of the p_{k1}'s and ρ_{k1}'s, each of which constitutes a total of M parameters, and the components of $\underset{\sim}{e}$ and $\underset{\sim}{h}$, which adds another six. The total number of unknowns, then, is (5M+6), which is the same as the number of parameters. The components of $\underset{\sim}{j}$ are not additional unknowns, because $\underset{\sim}{j}$ is a linear combination of the $\underset{\sim}{u}_k$'s.

We now note that each gas was assumed to obey an ideal gas law

$$\frac{p_{\ell o}}{N_{\ell o}} = \frac{m_\ell p_{\ell o}}{\rho_{\ell o}} = kT_{\ell o} \tag{16}$$

where k is Boltzmann's constant and $T_{\ell o}$ the zero-order absolute temperature of the ℓ^{th} constituent gas.

3. The Neutral-Electron-Ion Gas

We will now specialize to the three-fluid (neutral, positive ion, electron) gas, the homogeneous case of which was treated by Tanenbaum and Mintzer[1] and by Cronson and Clark[2]. In this case:

k = 1; neutral gas; subscript n replaces subscript 1 on all quantities;

$q_1 = q_n = 0$.

k = 2; ion gas, subscript i replaces subscript 2 on all quantities; $q_2 = +eZ$, where e is the electron charge* and Z the degree of ionization.

* Not to be confused with $\underset{\sim}{e}$, the electric field, or its components, although the same letter is used to represent both quantities.

$k = 3$; electron gas; subscript e replaces subscript 3; $q_3 = -e$.

We now invoke the following definitions and assumptions, most of which were used by Cronson and Clark[2],

Definitions:

$p_p \equiv$ partial pressure of plasma (electron plus ion gas) $= p_i + p_e$

$\rho_p \equiv$ mass density of plasma $= \rho_i + \rho_e = N_i m_i + N_e m_e$

$\underset{\sim}{u}_p \equiv$ average plasma velocity $= \dfrac{\rho_{io}\underset{\sim}{u}_i + \rho_{eo}\underset{\sim}{u}_e}{\rho_{io} + \rho_{eo}}$

$\underset{\sim}{j} \equiv eZN_{io}\underset{\sim}{u}_i - eN_{eo}\underset{\sim}{u}_e$

$\xi \equiv \dfrac{Zm_e}{m_i}$

Assumptions:

$$\frac{\rho_{io}}{m_i} Z = N_{io} Z = N_{eo} = \frac{\rho_{eo}}{m_e} \quad ; \text{electrical neutrality (in zero-order)} \tag{17}$$

from which it follows, through the definition of $\underset{\sim}{j}$ above, that

$$\underset{\sim}{j} = \frac{e\rho_{eo}}{m_e}(\underset{\sim}{u}_i - \underset{\sim}{u}_e) \tag{18}$$

$\dfrac{Zm_e}{m_i} \ll m_i$, or $\xi \ll 1$; (Electron mass is extremely small compared to ion mass and Z is a number of order unity) (19)

from which it follows, through the above definition of $\underset{\sim}{u}_p$, that

$$\frac{\underset{\sim}{u}_p \approx \underset{\sim}{u}_i + \dfrac{N_{eo}}{N_{io}}\dfrac{m_e}{m_i}\underset{\sim}{u}_e}{1 + \dfrac{N_{eo}}{N_{io}m_i}} = \frac{\underset{\sim}{u}_i + \xi\,\underset{\sim}{u}_e}{1+\xi} \approx \underset{\sim}{u}_i + \xi\,\underset{\sim}{u}_e \tag{20}$$

From the ideal gas law,

$$\frac{T_{eo}}{T_{io}} = \frac{p_{eo}N_{io}}{p_{io}N_{eo}} = 1 \tag{21}$$

i.e. T_{eo} and T_{io} the zero-order electron and ion temperature are assumed equal, or equivalently, thermal equilibrium between electrons and ions is assumed.

We now note that (see eqn. 11)

$$\frac{p_e}{\xi p_1} = \frac{N_{eo} m_e \overline{|\underset{\sim}{u}_e - \bar{\underset{\sim}{u}}_e|^2}/3}{\xi N_{io} m_i \overline{|\underset{\sim}{u}_i - \bar{\underset{\sim}{u}}_i|^2}/3} = \frac{\overline{|\underset{\sim}{u}_e - \bar{\underset{\sim}{u}}_e|^2}}{\overline{|\underset{\sim}{u}_i - \bar{\underset{\sim}{u}}_i|^2}} \tag{22}$$

and assume that the electrons, being many times lighter than the ions, attain much higher rms fluctuation velocities. This assumption results in the statement that

$$p_e >> \xi p_i \quad . \tag{23}$$

Also, from (17) and (21)

$$\frac{p_{eo}}{p_{io}} = \frac{N_{eo}}{N_{io}} = Z \quad . \tag{24}$$

Note that, from (24)

$$\frac{c_{seo}^2}{c_{sio}^2} = \frac{p_{eo}}{p_{io}} \frac{\rho_{io}}{\rho_{eo}} = \frac{Z}{\xi} >> 1 \quad . \tag{25}$$

We now define a "plasma sound velocity" c_{spo} by

$$c_{spo}^2 = \frac{\gamma p_{po}}{\rho_{po}} = \frac{c_{sio}^2 \rho_{io} + c_{seo}^2 \rho_{eo}}{\rho_{io} + \rho_{eo}} = \frac{\gamma(p_{io} + p_{eo})}{(\rho_{io} + \rho_{eo})} =$$

$$\frac{c_{sio}^2(1 + Z)}{(1 + \xi)} \approx c_{sio}^2(1 + Z) \tag{26}$$

and note that

$$\frac{c_{spo}^2}{c_{sno}^2} = \frac{p_{po}}{p_{no}} \cdot \frac{1}{\alpha} \tag{27}$$

where

$$\alpha \equiv \frac{\rho_{po}}{\rho_{no}} = \text{fractional ionization.}$$

By solving (18) and (20) simultaneously for u_i and u_e, with the aid of (19), we obtain

$$\underset{\sim}{u}_i \approx \underset{\sim}{u}_p + \frac{\underset{\sim}{j}}{\eta \rho_{po}} \tag{28}$$

$$\underset{\sim}{u}_e \approx \underset{\sim}{u}_p - \frac{\underset{\sim}{j}}{\eta \xi \rho_{po}} \tag{29}$$

where $\eta \equiv \frac{e}{m_e}$.

We now write the equations I-k, II-k, III-k for the case under consideration

$$\text{I'-1 } \rho_{no}\left(\frac{\partial \underset{\sim}{u}}{\partial t}\right)_n = -\nabla p_{n1} + \rho_{n1}\nabla\phi_g + \rho_{no}\nu_{ni}(\underset{\sim}{u}_n - \underset{\sim}{u}_i) + \rho_{no}\nu_{ne}(\underset{\sim}{u}_n - \underset{\sim}{u}_e)$$
$$\text{-2 } i \qquad\qquad i \qquad\quad i \qquad\quad i \quad in \quad i \quad n \qquad i \quad ie \quad i \quad e$$
$$\text{-3 } e \qquad\qquad e \qquad\quad e \qquad\quad e \quad en \quad e \quad n \qquad e \quad ei \quad e \quad i$$

$$+ \begin{pmatrix} 0 \\ +eZN_{io} \\ -eN_{eo} \end{pmatrix} \underset{e}{\overset{i}{(}} \underset{\sim}{e} + \mu_o \underset{\sim}{u}_n \times \underset{\sim}{h}_o \underset{e}{\overset{i}{)}}$$

$$\text{II'-1 } \nabla \cdot (\rho_{no} \underset{\sim}{u}_n) + \left(\frac{\partial p}{\partial t}\right)_{n1} = 0$$
$$\text{2 } \qquad i \quad i \qquad\quad i$$
$$\text{3 } \qquad e \quad e \qquad\quad e$$

$$\text{III'-1 } \left(\frac{\partial p}{\partial t}\right)_{n1} - c^2_{sno}\left(\frac{\partial \rho}{\partial t}\right)_{n1} = -\underset{\sim}{u}_n \cdot [\nabla p_{no} - c^2_{sno}\nabla \rho_{no}]$$
$$\text{-2 } \qquad i \qquad\quad i \qquad\quad i \qquad\quad i \qquad\quad i$$
$$\text{-3 } \qquad e \qquad\quad e \qquad\quad e \qquad\quad e \qquad\quad e$$

The following operations are now performed:

(A) Add I'-2 to I'-3

(B) Subtract (I'-3 multiplied by $\frac{e}{m_e}$) from (I'-2 multiplied by $\frac{eZ}{m_i}$)

(C) Add II'-2 to II'-3

(D) Subtract (II'-3 multiplied by $\frac{e}{m_e}$) from (II'-2 multiplied by $\frac{eZ}{m_i}$)

(E) Add III'-2 to III'-3

(F) Subtract (III'3 multiplied by $\frac{e}{m_e}$) from III'2 multiplied by $\frac{eZ}{m_i}$)

Performance of these operations and use of the definitions of ρ_p, p_p, $\underset{\rightharpoonup}{u}_p$ and $\underset{\rightharpoonup}{j}$ given above, eq. (17), together with eqs. (13) and eqs. (17), (19), (21), (23), (25), (26), (28) and (29), results in the following set of equations describing the system.

$$\text{I''-n, } \rho_{no} \frac{\partial \underset{\rightharpoonup}{u}_n}{\partial t} = -\nabla p_{n1} + \rho_{n1} \nabla \phi_g + \alpha \rho_{no} (\nu_{in} + \xi \nu_{en}) (\underset{\rightharpoonup}{u}_n - \underset{\rightharpoonup}{u}_p) + \left(\frac{\nu_{en} - \nu_{in}}{\eta}\right) \underset{\rightharpoonup}{j}$$

(Motion equation for neutral gas)

$$\text{I''-p; } \rho_{po} \frac{\partial \underset{\rightharpoonup}{u}_p}{\partial t} = -\nabla p_{p1} + \rho_{p1} \nabla \phi_g + \underset{\rightharpoonup}{j} \times \underset{\rightharpoonup}{B}_o + \rho_{po} (\nu_{in} + \xi \nu_{en}) (\underset{\rightharpoonup}{u}_p - \underset{\rightharpoonup}{u}_n)$$

$$+ \left(\frac{\nu_{in} - \nu_{en}}{\eta}\right) \underset{\rightharpoonup}{j}$$

(where $\underset{\rightharpoonup}{B}_o$ has replaced $\mu_o \underset{\rightharpoonup}{h}_o$)

$$\text{I''-0} \frac{\partial \underset{\rightharpoonup}{j}}{\partial t} = -\eta \nabla p_{\ell 1} + \frac{e^2 N_{eo}}{m_e} \underset{\rightharpoonup}{e}$$

$$+ \eta \rho_{po} \xi (\nu_{en} - \nu_{in}) (\underset{\rightharpoonup}{u}_n - \underset{\rightharpoonup}{u}_p) + (\nu_{en} + \xi \nu_{in} + \nu_{ei}) \underset{\rightharpoonup}{j} - i_z (g \rho_{c1})$$

$$+ \frac{e^2 N_{eo}}{m_e} \underset{\rightharpoonup}{u}_p \times \underset{\rightharpoonup}{B}_o - \frac{e^2 N_{eo}}{m_e \eta \rho_{po} \xi} (\underset{\rightharpoonup}{j} \times \underset{\rightharpoonup}{B}_o)$$

where $p_{\ell 1} \equiv (\xi p_{i1} - p_{e1})$

(generalized Ohm's Law for plasma)

$$\text{II''-n } \nabla \cdot (\rho_{no} \underset{\rightharpoonup}{u}_n) + \frac{\partial \rho_{n1}}{\partial t} = 0 \text{ (continuity for neutral gas)}$$

$$\text{II''-p: } \nabla \cdot (\rho_{po} \underset{\rightharpoonup}{u}_p) + \frac{\partial \rho_{p1}}{\partial t} = 0 \text{ (continuity for plasma)}$$

$$\text{II''-0: } \nabla \cdot \underset{\rightharpoonup}{j} + \frac{\partial \rho_{c1}}{\partial t} = 0 \text{ (electrical equation of continuity or charge conservation)}$$

where $\rho_{c1} \equiv eZN_{i1} = eN_{e1}$ = perturbed or first order charge density, not necessarily zero

$$\text{III''-n:}\quad \frac{\partial \rho_{n1}}{\partial t} - c_{sno}^{\,2}\,\frac{\partial \rho_{n1}}{\partial t} = -\,\underset{\sim}{u}_n \cdot \left[\nabla p_{no} - c_{sno}^{\,2}\,\nabla \rho_{no}\right]$$

(Adiabatic state for neutral gas)

$$\text{III''-p:}\quad \frac{\partial p_{p1}}{\partial t} - c_{spo}^{\,2}\,\frac{\partial \rho_{p1}}{\partial t} = -\,\underset{\sim}{u}_p \cdot \left[\nabla p_{po} - c_{spo}^{\,2}\nabla \rho_{po}\right] + \frac{Z\underset{\sim}{j}}{\xi\eta\rho_{po}(1+Z)} \cdot \left[\nabla p_{po} - c_{spo}^{\,2}\nabla \rho_{po}\right] - \frac{Z}{(1+Z)}\,\frac{c_{spc}^{\,2}}{\xi\eta}\,\frac{\partial \rho_{c1}}{\partial t}$$

(Adiabatic state for plasma)

$$\text{III''-0:}\quad \frac{\partial p_{\ell 1}}{\partial t} + \frac{Zc_{spo}^{\,2}}{(1+Z)}\,\frac{\partial \rho_{p1}}{\partial t} = \frac{Z}{(1+Z)}\,\underset{\sim}{u}_p \cdot \left[\nabla p_{po} - c_{spo}^{\,2}\nabla \rho_{po}\right] - \frac{Z\underset{\sim}{j}}{(1+Z)\eta\xi\,\rho_{po}} \cdot \left[\nabla p_{po} - c_{spo}^{\,2}\nabla \rho_{po}\right] + \frac{c_{spo}^{\,2}}{\xi\eta}\,\frac{\partial \rho_{c1}}{\partial t}$$

$$\text{IV''}:\quad \nabla \times \underset{\sim}{e} = -\,\mu_o\,\frac{\partial \underset{\sim}{h}_1}{\partial t}$$

$$\text{V''}:\quad \nabla \times \underset{\sim}{h}_1 = \underset{\sim}{j} + \epsilon_o\,\frac{\partial \underset{\sim}{e}}{\partial t}\quad .$$

Note that there are other equally satisfactory ways to describe the three-fluid (electron-ion-neutral) gas, such as the retention of the original equations I'-1, 2, 3, II'-1, 2, 3, III'-1, 2, 3, wherein the unknowns are $u_{n,i,e}$, $p_{n,i,e}$, --- etc., instead of $u_{n,p}$, $p_{n,p}$, j, as in the present formulation. Tanenbaum and Mintzer[1] use the (n, i, e,) form of the equations, while Cronson and Clark[2], following Spitzer[6], use the (n, p) form. The latter approach conveniently lumps the electron and ion gas equations into a single plasma equation, and the current density j appears in the Maxwell equations as a single unknown vector instead of a linear combination of the vectors u_i and u_e. However, there is no real

difference in principle or in mathematical difficulty between these two methods of approach. Both leave the same number of equations and unknowns, but the equations appear different in form and the unknowns are changed.

Still another method, that used by Cowling[7], lumps the three gases together into a composite neutral-plus-plasma gas, defining variables for that composite gas. The approach illuminates different physical parameters than the present approach, but does not change the degree of difficulty in solving the equations. Moreover, solutions obtained with one of these approaches can always be used to obtain the solutions that would have been obtained by another, since the former sets of solutions are linear combinations of the latter.

4. The Vertically Non-uniform Neutral-Electron-Ion Gas

We now further specialize to the case where the ambient three-fluid gas has parameters that are constant in time and horizontally uniform, but vary with vertical position. To treat this case, we use a right-handed cartesian coordinate system (x, y, z) and look for plane wave solutions in the time t and the horizontal coordinates x and y, reducing the problem to a system of ordinary differential equations in the vertical coordinate z. Thus each first order (unknown) parameter, designated generically by u, has a complex Fourier solution of the form

$$u(x, y, z, t) = U(k_x, k_y, z, \omega) e^{-i(\omega t - k_x x - k_y y)} , \tag{30}$$

(i.e. the wave solution corresponding to a given lower case letter is denoted by its corresponding capital letter, except for the solution corresponding to ρ, which is denoted by $\mathcal{R}$) where k_x and k_y are in general complex while the angular frequency ω is real.

We define a horizontal wave vector k_h by

$$\underset{\rightarrow}{k}_h = \underset{\rightarrow}{i}_x k_x + \underset{\rightarrow}{i}_y k_y \tag{31'}$$

where ($\underset{\rightarrow}{i}_x$, $\underset{\rightarrow}{i}_y$, $\underset{\rightarrow}{i}_z$) are unit vectors in the x, y, z directions resulting in the following forms for eqs. I'' through V''.

$$\text{I}'''\text{-n};\ i\omega\rho_{no}\underset{\rightarrow}{U}_n = -ik_h P_{n1} - \underset{\rightarrow}{i}_z \frac{\partial P_{n1}}{\partial z} + \underset{\rightarrow}{i}_z \mathcal{R}_{n1} \frac{\partial \phi_g}{\partial z}$$

$$+ \alpha\rho_{no}(\nu_{in} + \xi\nu_{en})(\underset{\rightarrow}{U}_n - \underset{\rightarrow}{U}_p) + \left(\frac{\nu_{en} - \nu_{in}}{\eta}\right)\underset{\rightarrow}{J}$$

$$\text{I}'''\text{-p};\ i\omega\rho_{po}\underset{\rightarrow}{U}_p = -ik_h P_{p1} - \underset{\rightarrow}{i}_z \frac{\partial P_{p1}}{\partial z} + \underset{\rightarrow}{i}_z \mathcal{R}_{p1} \frac{\partial \phi_g}{\partial z}$$

$$+ \rho_{po}(\nu_{in} + \xi\nu_{en})(\underset{\rightarrow}{U}_p - \underset{\rightarrow}{U}_n) + \left(\frac{\nu_{in} - \nu_{en}}{\eta}\right)\underset{\rightarrow}{J} + \underset{\rightarrow}{J} \times \underset{\rightarrow}{B}_o$$

$$\text{I}'''\text{-0};\ (-i\omega - \nu_{en} - \xi\nu_{in} - \nu_{ei})\underset{\rightarrow}{J} + \frac{e^2 N_{eo}}{m_e \nu_{eo} \xi} \underset{\rightarrow}{J} \times \underset{\rightarrow}{B}_o =$$

$$\frac{e^2 N_{eo}}{m_e} \underset{\rightarrow}{U}_p \times \underset{\rightarrow}{B}_o + \frac{e^2 N_{eo}}{m_e} \underset{\rightarrow}{E} - ik_h P_{\ell 1} - \underset{\rightarrow}{i}_z \frac{\partial P_{\ell 1}}{\partial z}$$

$$+ \eta\rho_{po}\xi(\nu_{en} - \nu_{in})(\underset{\rightarrow}{U}_n - \underset{\rightarrow}{U}_p) - \underset{\rightarrow}{i}_z(g\mathcal{R}_{c1})$$

$$\text{II}'''\text{-n};\ i\rho_{no}\underset{\rightarrow}{k}_h \cdot \underset{\rightarrow}{U}_n + \rho_{no}\frac{\partial U_{nz}}{\partial z} + U_{nz}\frac{\partial \rho_{no}}{\partial z} - i\omega\mathcal{R}_{n1} = 0$$

$$\text{II}'''\text{-p};\ i\rho_{po}\underset{\rightarrow}{k}_h \cdot \underset{\rightarrow}{U}_p + \rho_{po}\frac{\partial U_{pz}}{\partial z} + U_{pz}\frac{\partial \rho_{po}}{\partial z} - i\omega\mathcal{R}_{p1} = 0$$

$$\text{II}'''\text{-0};\ i\underset{\rightarrow}{k}_h \cdot \underset{\rightarrow}{J} + \underset{\rightarrow}{i}_z \frac{\partial J_z}{\partial z} - i\omega\mathcal{R}_{c1} = 0$$

$$\text{III}'''\text{-n};\ -i\omega P_{n1} + c_{sno}^{2}\, i\omega \mathcal{R}_{n1} = -U_{nz}\left[\frac{\partial p_{no}}{\partial z} - c_{sno}^{2}\frac{\partial \rho_{no}}{\partial z}\right]$$

$$\text{III}'''\text{-p};\ i\omega P_{p1} + c_{spo}^{2} i\omega \mathcal{R}_{p1} = -U_{pz}\left[\frac{\partial p_{po}}{\partial z} - c_{spo}^{2}\frac{\partial \rho_{po}}{\partial z}\right]$$

$$+ \frac{Z}{(1+Z)}\frac{J_z}{\xi\eta\rho_{po}}\left[\frac{\partial p_{po}}{\partial z} - c_{spo}^{2}\frac{\partial \rho_{po}}{\partial z}\right] + \frac{Z}{(1+Z)}\frac{c_{spo}^{2}}{\xi\eta} i\omega \mathcal{R}_{c1}$$

$$\text{III}'''\text{-0};\ -i\omega P_{\ell 1} - \frac{Zc_{spo}^{2}}{(1+Z)} i\omega \mathcal{R}_{p1} = -\frac{ZJ_z}{(1+Z)\eta\xi\rho_{po}}\left[\frac{\partial p_{\rho o}}{\partial z} - c_{spo}^{2}\frac{\partial \rho_{po}}{\partial z}\right]$$

$$+ \frac{Z}{(1+Z)} U_{pz}\left[\frac{\partial p_{po}}{\partial z} - c_{spo}^{2}\frac{\partial \rho_{po}}{\partial z}\right] - \frac{c_{spo}^{2}}{\xi\eta} i\omega \mathcal{R}_{c1}$$

$$\text{IV}'''';\ i\underset{\sim}{k}_h \times \underset{\sim}{E} + \underset{\sim}{i}_x\left(-\frac{\partial E_y}{\partial z}\right) + \underset{\sim}{i}_y\left(\frac{\partial E_x}{\partial z}\right) = i\omega\mu_o \underset{\sim}{H}_1$$

$$\text{V}'''';\ i\underset{\sim}{k}_h \times \underset{\sim}{H}_1 + \underset{\sim}{i}_x\left(-\frac{\partial H_{iy}}{\partial z}\right) + \underset{\sim}{i}_y\left(\frac{\partial H_{ix}}{\partial z}\right) = -i\omega\epsilon_o \underset{\sim}{E} + \underset{\sim}{J} \quad .$$

It is convenient to normalize the quantities in the equations I'''-n through V'''' such that each quantity in the equations is dimensionless. To this end we denote each normalized parameter by its symbol with a karet above it and invoke the following definitions:

$$\hat{\nu}_{en,in,ei} \equiv \frac{\nu_{en,in,ei}}{\omega}$$

$$\hat{c}_{sko} \equiv \frac{c_{sko}}{c} \quad ; k = n,\ p,\ i,\ e$$

$$c = \text{light velocity in vacuo} \equiv \frac{1}{\sqrt{\mu_o \epsilon_o}}$$

$$\omega_{ce} \equiv \eta\mu_o H_o = \text{electron cyclotron frequency}$$

$$\hat{\omega}_{ce} \equiv \frac{\omega_{ce}}{\omega}$$

$$k_{\rho n, \rho p} \equiv \frac{1}{\rho_{no, po}} \frac{\partial \rho_{no, po}}{\partial z}$$

$$\kappa_{ck} \equiv \frac{1}{c_{sko}} \frac{\partial c_{sko}}{\partial z} \; ; \; k = n, \; p, \; i, \; e$$

$k_o \equiv$ free space electromagnetic wave number $= \frac{\omega}{c} = \omega \sqrt{\mu_o \epsilon_o}$

$$\hat{k}_h \equiv \frac{k_h}{k_o} \; ; \; \hat{k}_x \equiv \frac{k_x}{k_o}$$

$\omega_p \equiv$ electron plasma resonant frequency $= \sqrt{\frac{\eta^2 \rho_{po} \xi}{\epsilon_o}}$

$$\hat{\omega}_p \equiv \frac{\omega_p}{\omega}$$

$$\hat{\underset{\rightarrow}{U}}_{n, p} \equiv \frac{\underset{\rightarrow}{U}_{n, p}}{c}$$

$$\hat{P}_{n1, p1} \equiv \frac{P_{n1, p1}}{p_{no, po}}$$

$$\hat{\mathcal{R}}_{n1, p1} \equiv \frac{\mathcal{R}_{n1, p1}}{\rho_{no, po}}$$

$$\hat{P}_{\ell 1} \equiv \frac{P_{\ell 1}}{p_{po}}$$

$$\hat{z} \equiv k_o z$$

$$\hat{\underset{\rightarrow}{J}} \equiv \frac{\underset{\rightarrow}{J}}{\eta \xi \rho_{po} c}$$

$$\hat{\underset{\rightarrow}{E}} \equiv \left(\frac{\omega \epsilon_o}{\eta \xi \rho_{po} c} \right) \underset{\rightarrow}{E}$$

$$\hat{\underset{\sim}{B}}_1 = \mu_o \hat{\underset{\sim}{H}}_1 \equiv \left(\frac{\omega \epsilon_o \mu_o}{\eta \xi \rho_{po}} \right) \underset{\sim}{H}_1 = \left(\frac{\omega \epsilon_o}{\eta \xi \rho_{po}} \right) \underset{\sim}{B}_1$$

$$\underset{\sim}{n}_B \equiv \frac{\underset{\sim}{H}_o}{|\underset{\sim}{H}_o|} \equiv \text{unit vector in direction of } \underset{\sim}{H}_o$$

$$\hat{\mathcal{R}}_{c1} \equiv \frac{\mathcal{R}_{c1}}{\eta \xi \rho_{po}} \quad .$$

We also assume that:

$$\nabla \phi_g = \underset{\sim}{g} = - g \underset{\sim}{i}_z = - \hat{g} \omega c \underset{\sim}{i}_z \tag{32}$$

where g ≡ acceleration to gravity = 9.8 $\frac{m}{sec^2}$ and $\hat{g} \equiv \frac{g}{\omega c}$.

and conclude from (23) that

$$p_\ell \equiv \xi p_i - p_e \approx - p_e \tag{33}$$

i.e. the parameter p_ℓ, which enters into later discussions. is approximately equated to the negative of electron pressure.

In terms of the normalized parameters and with the aid of the assumption (32), we have

$$\hat{I}\text{-n: } (1 + i\alpha (\hat{\nu}_{in} + \xi \hat{\nu}_{en})) \hat{\underset{\sim}{U}}_n - i\alpha (\hat{\nu}_{in} + \xi \hat{\nu}_{en}) \hat{\underset{\sim}{U}}_p + \underset{\sim}{i}_z (i\hat{g}) \hat{\mathcal{R}}_{n1}$$

$$+ \left[\underset{\sim}{i}_z \left(\frac{i\hat{c}_{sno}^2}{\gamma} \right) (\hat{k}_{\rho n} + 2\hat{k}_{cn}) - \underset{\sim}{k}_h \left(\frac{\hat{c}_{sno}^2}{\gamma} \right) \right] \hat{P}_{n1}$$

$$+ \underset{\sim}{i}_z \left(\frac{i\hat{c}_{sno}^2}{\gamma} \right) \frac{\partial \hat{P}_{n1}}{\partial \hat{z}} + i\alpha \xi (\hat{\nu}_{en} - \hat{\nu}_{in}) \hat{\underset{\sim}{J}} = 0$$

$$\hat{I}\text{-p: } (1 + i(\hat{\nu}_{in} + \xi \hat{\nu}_{en})) \hat{\underset{\sim}{U}}_p - i(\hat{\nu}_{in} + \xi \hat{\nu}_{en}) \hat{\underset{\sim}{U}}_n + \underset{\sim}{i}_z (i\hat{g}) \hat{\mathcal{R}}_{p1}$$

$$+ \left[\underset{\sim}{i}_z \left(\frac{i\hat{c}_{spo}^2}{\nu} \right) (\hat{k}_{\rho p} + 2\hat{k}_{cp}) - \underset{\sim}{k}_h \left(\frac{\hat{c}_{spo}^2}{\gamma} \right) \right] \hat{P}_{p1}$$

$$+ \underset{\sim}{i}_z \left(\frac{i\hat{c}_{spo}^2}{\gamma} \right) \frac{\partial \hat{P}_{p1}}{\partial \hat{z}} - i\xi (\hat{\nu}_{en} - \hat{\nu}_{in}) \hat{\underset{\sim}{J}} - \xi \hat{\omega}_{ce} (\hat{\underset{\sim}{J}} \times \underset{\sim}{n}_B) = 0$$

$$\hat{I}\text{-}0:\ (1 + i[\hat{\nu}_{en} + \xi\,\hat{\nu}_{in} + \hat{\nu}_{ei}])\,\hat{\underset{\sim}{J}} - i(\hat{\nu}_{en} - \hat{\nu}_{in})\,\hat{\underset{\sim}{U}}_p + i(\hat{\nu}_{en} - \hat{\nu}_{in})\,\hat{\underset{\sim}{U}}_n$$

$$+ \left[\underset{\sim}{i}_z \left(\frac{i\hat{c}_{spo}^{\,2}}{\gamma\xi}\right)[\hat{k}_{\rho p} + 2\hat{k}_{cp}] - \underset{\sim}{k}_h \left(\frac{\hat{c}_{spo}^{\,2}}{\gamma\xi}\right)\right]\hat{P}_{\ell 1} + \underset{\sim}{i}_z \left(\frac{i\hat{c}_{spo}^{\,2}}{\gamma\xi}\right)\frac{\partial \hat{P}_{\ell 1}}{\partial \hat{z}}$$

$$- i\hat{\omega}_p^{\,2}\,\hat{\underset{\sim}{E}} - i\hat{\omega}_{ce}(\hat{\underset{\sim}{U}}_p \times \underset{\sim}{n}_B) + i\hat{\omega}_{ce}(\hat{\underset{\sim}{J}} \times \underset{\sim}{n}_B) - \underset{\sim}{i}_z\left(\frac{ig\hat{\mathcal{R}}_{c1}}{\zeta}\right) = 0$$

$$\hat{II}\text{-}n:\ \underset{\sim}{\hat{k}}_h \cdot \hat{\underset{\sim}{U}}_n - i\hat{k}_{\rho n}\,\hat{U}_{nz} - i\,\frac{\partial \hat{U}_{nz}}{\partial \hat{z}} - \hat{\mathcal{R}}_{n1} = 0$$

$$\hat{II}\text{-}p:\ \underset{\sim}{\hat{k}}_h \cdot \hat{\underset{\sim}{U}}_p - i\hat{k}_{\rho p}\,\hat{U}_{pz} - i\,\frac{\partial \hat{U}_{pz}}{\partial \hat{z}} - \hat{\mathcal{R}}_{p1} = 0$$

$$\hat{II}\text{-}0:\ i\underset{\sim}{\hat{k}}_h \cdot \hat{\underset{\sim}{J}} + \hat{k}_{\rho p}\,\hat{J}_z + \frac{\partial \hat{J}_z}{\partial \hat{z}} - i\hat{\mathcal{R}}_{c1} = 0$$

$$\hat{III}\text{-}n:\ \hat{P}_{n1} - \gamma\hat{\mathcal{R}}_{n1} + i[\hat{k}_{\rho n}(1 - \gamma) + 2\hat{k}_{cn}]\,\hat{U}_{nz} = 0$$

$$\hat{III}\text{-}p:\ \hat{P}_{p1} - \gamma\hat{\mathcal{R}}_{p1} + i[\hat{k}_{\rho p}(1 - \gamma) + 2\hat{k}_{cp}]\,\hat{U}_{pz} - \frac{iZ}{(1+Z)}[\hat{k}_{\rho p}(1 - \gamma) + 2\hat{k}_{cp}]\,\hat{J}_z + \frac{\gamma Z\hat{\mathcal{R}}_{c1}}{(1+Z)} = 0$$

$$\hat{III}\text{-}0:\ \left(\frac{1+Z}{Z}\right)\hat{P}_{\ell 1} + \gamma\hat{\mathcal{R}}_{p1} + i\hat{J}_z[\hat{k}_{\rho p}(1-\gamma) + 2\hat{k}_{cp}] - i\hat{U}_{pz}[\hat{k}_{\rho p}(1-\gamma) + 2\hat{k}_{cp}] - \gamma\hat{\mathcal{R}}_{c1} = 0$$

$$\hat{IV}:\ i\underset{\sim}{\hat{k}}_h \times \hat{\underset{\sim}{E}} + \underset{\sim}{i}_x\left(-\frac{\partial \hat{E}_y}{\partial z} - \hat{k}_{\rho p}\hat{E}_y\right) + \underset{\sim}{i}_y\left(\frac{\partial \hat{E}_x}{\partial \hat{z}} + \hat{k}_{\rho p}\hat{E}_x\right) = i\hat{\underset{\sim}{B}}_1$$

$$\hat{V}:\ i\underset{\sim}{\hat{k}}_h \times \hat{\underset{\sim}{B}}_1 + \underset{\sim}{i}_z\left(-\frac{\partial \hat{B}_{1y}}{\partial \hat{z}} - \hat{k}_{\rho p}\hat{B}_{1y}\right) + \underset{\sim}{i}_y\left(\frac{\partial \hat{B}_{ix}}{\partial \hat{z}} + \hat{k}_{\rho p}\hat{B}_{ix}\right) = -i\hat{\underset{\sim}{E}} + \hat{\underset{\sim}{J}}$$

4.1. The Two-Dimensional Case

Consider the case where the source of the wave disturbance under study is a uniform infinitely long line-source along the y-axis and the ambient gas parameters are independent of x and y. With this kind of symmetry, the unknowns cannot be functions of y. We can therefore set $k_y = 0$ and the horizontal

propagation vector k_h becomes $i_x k_x$. We now resolve the vector equations $\hat{I}'''$, $\hat{IV}'''$ and $\hat{V}'''$ into component equations, assigning the appropriate symbols to denote component equations x, y and z.

$$\hat{\hat{I}}\text{-n-}\begin{matrix}x\\y\\z\end{matrix}: \; -\frac{i\gamma}{\hat{c}_{sno}^{\,2}}\left[1+i\alpha\left(\hat{\nu}_{in}+\xi\hat{\nu}_{en}\right)\right]\hat{U}_{n\begin{matrix}x\\y\\z\end{matrix}}+\begin{pmatrix} i\hat{k}_x \\ 0 \\ \hat{k}_{\rho n}+2k_{cn}\end{pmatrix}\hat{P}_{n1}+\begin{pmatrix}0\\0\\1\end{pmatrix}\frac{\partial\hat{P}_{n1}}{\partial\hat{z}}$$

$$-\left[\frac{\gamma\alpha}{\hat{c}_{sno}^{\,2}}\left(\hat{\nu}_{in}+\xi\hat{\nu}_{en}\right)\right]\hat{U}_{p\begin{matrix}x\\y\\z\end{matrix}}+\left[\frac{\gamma\alpha\xi}{\hat{c}_{sno}^{\,2}}\left(\hat{\nu}_{en}-\hat{\nu}_{in}\right)\right]\hat{J}_{\begin{matrix}x\\y\\z\end{matrix}}+\begin{pmatrix}0\\0\\\frac{\gamma\hat{g}}{\hat{c}_{sno}^{\,2}}\end{pmatrix}\hat{R}_{n1}=0$$

$$\hat{\hat{I}}\text{-p-}\begin{matrix}x\\y\\z\end{matrix}: \; -\frac{i\gamma}{\hat{c}_{spo}^{\,2}}\left[1+i\left(\hat{\nu}_{in}+\xi\hat{\nu}_{en}\right)\right]U_{p\begin{matrix}x\\y\\z\end{matrix}}+\begin{pmatrix} i\hat{k}_x \\ 0 \\ \hat{k}_{\rho p}+2\hat{k}_{cp}\end{pmatrix}\hat{P}_{p1}+\begin{pmatrix}0\\0\\1\end{pmatrix}\frac{\partial\hat{P}_{p1}}{\partial\hat{z}}$$

$$-\left[\frac{\gamma}{\hat{c}_{spo}^{\,2}}\left(\hat{\nu}_{in}+\xi\hat{\nu}_{en}\right)\right]\hat{U}_{n\begin{matrix}x\\y\\z\end{matrix}}-\left[\frac{\gamma\xi}{\hat{c}_{spo}^{\,2}}\left(\hat{\nu}_{en}-\hat{\nu}_{in}\right)\right]\hat{J}_{\begin{matrix}x\\y\\z\end{matrix}}+\begin{pmatrix}0\\0\\\frac{\gamma\hat{g}}{\hat{c}_{spo}^{\,2}}\end{pmatrix}\hat{R}_{p1}$$

$$+\frac{i\gamma\xi\,\hat{\omega}_{ce}}{\hat{c}_{spo}^{\,2}}\left(n_{Bz\begin{matrix}x\\y\end{matrix}}\hat{J}_{y\begin{matrix}z\\x\end{matrix}}-n_{By\begin{matrix}z\\x\end{matrix}}\hat{J}_{z\begin{matrix}x\\y\end{matrix}}\right)=0$$

$$\hat{\hat{I}}\text{-0-}\begin{matrix}x\\y\\z\end{matrix}: \; -\frac{i\gamma\xi}{\hat{c}_{spo}^{\,2}}\left[1+i\left(\hat{\nu}_{en}+\xi\hat{\nu}_{in}+\hat{\nu}_{ei}\right)\right]\hat{J}_{\begin{matrix}x\\y\\z\end{matrix}}-\left[\frac{\gamma\xi}{\hat{c}_{spo}^{\,2}}\left(\hat{\nu}_{en}-\hat{\nu}_{in}\right)\right]\hat{U}_{p\begin{matrix}x\\y\\z\end{matrix}}+\begin{pmatrix}0\\0\\\frac{\gamma\hat{g}\xi}{\hat{c}_{spo}^{\,2}}\end{pmatrix}\hat{R}_{c1}$$

$$+\left[\frac{\gamma\xi}{\hat{c}_{spo}^{\,2}}\left(\hat{\nu}_{en}-\hat{\nu}_{in}\right)\right]\hat{U}_{n\begin{matrix}x\\y\\z\end{matrix}}+\begin{pmatrix} i\hat{k}_x \\ 0 \\ \hat{k}_{\rho p}+2\hat{k}_{cp}\end{pmatrix}\hat{P}_{\ell 1}+\begin{pmatrix}0\\0\\1\end{pmatrix}\frac{\partial\hat{P}_{\ell 1}}{\partial\hat{z}}-\left(\frac{\gamma\xi\,\hat{\omega}_p^{\,2}}{\hat{c}_{spo}^{\,2}}\right)\hat{E}_{\begin{matrix}x\\y\\z\end{matrix}}$$

$$+\frac{\gamma\xi\,\hat{\omega}_{ce}}{\hat{c}_{spo}^{\,2}}\left(n_{Bz\begin{matrix}x\\y\end{matrix}}\hat{J}_{y\begin{matrix}z\\x\end{matrix}}-n_{By\begin{matrix}z\\x\end{matrix}}J_{z\begin{matrix}x\\y\end{matrix}}\right)-\frac{\gamma\xi\,\hat{\omega}_{ce}}{\hat{c}_{spo}^{\,2}}\left(n_{Bz\begin{matrix}x\\y\end{matrix}}\hat{U}_{py\begin{matrix}z\\x\end{matrix}}-n_{By\begin{matrix}z\\x\end{matrix}}\hat{U}_{pz\begin{matrix}x\\y\end{matrix}}\right)=0$$

$$\hat{\text{II}}\text{-n:}\quad i\hat{k}_x \hat{U}_{nx} + \hat{k}_{\rho n} \hat{U}_{nz} + \frac{\partial \hat{U}_{nz}}{\partial \hat{z}} - i\hat{\mathcal{R}}_{n1} = 0$$

$$\hat{\text{II}}\text{-p:}\quad i\hat{k}_x \hat{U}_{px} + \hat{k}_{\rho p} \hat{U}_{pz} + \frac{\partial \hat{U}_{pz}}{\partial \hat{z}} - i\hat{\mathcal{R}}_{p1} = 0$$

$$\hat{\text{II}}\text{-0:}\quad i\hat{k}_x \hat{J}_x + \hat{k}_{\rho p} \hat{J}_z + \frac{\partial \hat{J}_z}{\partial \hat{z}} - i\hat{\mathcal{R}}_{c1} = 0$$

$$\hat{\text{III}}\text{-n:}\quad \hat{P}_{n1} - \gamma\hat{\mathcal{R}}_{n1} + i[\hat{k}_{\rho n}(1-\gamma) + 2\hat{k}_{cn}]\,\hat{U}_{nz} = 0$$

$$\hat{\text{III}}\text{-p:}\quad \hat{P}_{p1} - \gamma\hat{\mathcal{R}}_{p1} + i[\hat{k}_{\rho p}(1-\gamma) + 2\hat{k}_{cp}]\,\hat{U}_{pz} - \frac{iZ}{(1+Z)}[\hat{k}_{\rho p}(1-\gamma) + 2\hat{k}_{cp}]\,\hat{J}_z + \frac{\gamma Z\hat{\mathcal{R}}_{c1}}{(1+Z)} = 0$$

$$\hat{\text{III}}\text{-0:}\quad \left(\frac{1+Z}{Z}\right)\hat{P}_{\ell 1} + \gamma\hat{\mathcal{R}}_{p1} + i[\hat{k}_{\rho p}(1-\gamma) + 2\hat{k}_{cp}]\,\hat{J}_z - i[\hat{k}_{\rho p}(1-\gamma) + 2\hat{k}_{cp}]\,\hat{U}_{pz} - \gamma\hat{\mathcal{R}}_{c1} = 0$$

$$\hat{\text{IV}}\text{-x:}\quad -\frac{\partial \hat{E}_y}{\partial \hat{z}} - \hat{k}_{\rho p} \hat{E}_y - i\hat{B}_{ix} = 0$$

$$\hat{\text{IV}}\text{-y:}\quad -i\hat{k}_x \hat{E}_z + \frac{\partial \hat{E}_x}{\partial \hat{z}} + \hat{k}_{\rho p} \hat{E}_x - i\hat{B}_{1y} = 0$$

$$\hat{\text{IV}}\text{-z:}\quad +\hat{k}_x E_y - B_{1z} = 0$$

$$\hat{\text{V}}\text{-x:}\quad -\frac{\partial \hat{B}_{1y}}{\partial \hat{z}} - \hat{k}_{\rho p} \hat{B}_{1y} + i\hat{E}_x - \hat{J}_x = 0$$

$$\hat{\text{V}}\text{-y:}\quad -i\hat{k}_x \hat{B}_{1z} + \frac{\partial \hat{B}_{1x}}{\partial \hat{z}} + \hat{k}_{\rho p} \hat{B}_{1x} + i\hat{E}_y - \hat{J}_y = 0$$

$$\hat{\text{V}}\text{-z:}\quad i\hat{k}_x \hat{B}_{1y} + i\hat{E}_z - \hat{J}_z = 0$$

4.1.1. Matrix Formulation

Eqns. $\hat{\hat{I}}$-n-x through $\hat{\hat{V}}$-z constitute a set of 21 coupled equations in 21 unknowns, of which 11 are purely algebraic and 10 are first order differential. To solve them by matrix methods, we first express them in the form

$$\frac{du(z)}{dz} = A'(z)\, u(z) + B'(z)\, v(z) \tag{34}$$

$$0 = C'(z)\, u(z) + D'(z)\, v(z) \tag{35}$$

where $A'(z)$, $B'(z)$, $C'(z)$ and $D'(z)$ are matrices consisting of known elements composed of certain coefficients of our equation system, while $u(z)$ and $v(z)$ are vectors whose elements are the unknown parameters. The matrices A', B', C' and D' and the vectors of u and v are shown in Tables 1 and 2.

To cast Eqns. (34) and (35) into the desired form, we perform the following matrix operations:

$$(D')^{-1}(C'u + D'v) = 0 = (D')^{-1} C'u + v \ ;$$

or

$$v = - (D')^{-1} C'u \tag{36}$$

$$\frac{du}{dz} = A'u + B' [- \{(D')^{-1} C'\}u] = [A' - B'(D')^{-1}C']u \tag{37}$$

or more concisely

$$\frac{du}{dz} = Au \tag{37}'$$

where

$$A \equiv A' - B'(D')^{-1}C' \ .$$

Note that the operation indicated in matrix notation by Eq. (36) is that of solving the set of 11 simultaneous equations for the elements of v, considered unknown, in terms of those of u, considered known. The next step represents the subsequent operation of substituting these solutions of (34) into (35) in order to obtain a system of coupled first order differential equations of the form (37).

4.1.2. Analytical Methods of Solution

The theory of systems of the class (37) is discussed in both the purely mathematical literature of differential equations[8, 9, 10, 11, 12] and in the literature of modern generalized systems and automatic control theory [13, 14, 15, 16]. In particular, the system (37) with constant A elements is the foundation of the "state-space" approach to modern automatic control theory[15, 16]. The literature was reviewed in an effort to find techniques for solution of the system (37) that apply to matrices A(z) with z-dependent elements. If these elements are independent of z, or, even if z-dependent, if they have the property of self-commutativity, i.e.

$$A(z_1)\,A(z_2) = A(z_2)\,A(z_1) \tag{38}$$

then solutions can be obtained in closed form[13, 14]; the problem degenerates into that of finding eigenvalues of the matrix $\int A(z)dz$. If A(z) does not have the property (38), then each of the elements may be separable into a major part that fulfills (38) and a "small perturbation" part that does not. The system is then solved with the assumption that the entire matrix is the self-commutative part, and this solution is used as the zero-order term of a perturbation solution for the more realistic problem involving the non self-

commutative matrix. The difficulty with this approach is that it is very difficult to find a meaningful separation of A(z) into self-commutative and non self-commutative parts.

Another possible analytical technique applies to periodic elements of A(z)[8, 9, 10, 11]. If the z-range is divided into strata, then in each stratum the elements A(z) could be considered periodic with a period equal to the size of the stratum. General theory exists on the periodic coefficient case[9], but its implementation requires knowledge of the "fundamental solutions" of the system, which is not easily attainable, or what amounts to a perturbation around the constant coefficient case, which may be very inaccurate unless the coefficients are nearly constant or unless many terms of a Fourier series solution are invoked.

Another possibility that was studied is the straightforward diagonalization of A(z) which effectively uncouples the differential equations (37) and transforms them into a set of first order differential equations in single variables, which are trivially simple to solve. The difficulty with this approach is that, because of the z variation of the elements of A(z), the diagonalization equation involves a "Liapounov transformation"[11] and takes the form

$$(T^{-1} A T - T^{-1} \frac{dT}{dz})_{ik} = \delta_{ik} B_{ik} \tag{39}$$

where δ_{ik} is the Kronecker delta and B_{ik} the i^{th} element of a matrix.

To find the transformation matrix T(z) that diagonalized A(z), it would be necessary to solve a set of first order coupled ordinary differential equations that may be, in the general case, as complicated as the original set; thus the problem is not reduced to the purely algebraic one of find the roots of the characteristic equation, as it would if A(z) were independent of z.

Still another analytical technique is the direct use of the "matricant series" [11, 12], which evolves naturally from an iterative solution of the equation system (37) and which is known to converge to the exact solution. In certain cases, e.g. where the matrix elements can be represented by the first two or three terms of a power series in z or by a constant term plus two or three exponential or Fourier terms, we might obtain a series whose convergence is fast enough to offer a possibility of practical solution. The actual implementation of this method to obtain useful solutions requires a high-speed computer. In the case of self-commutative matrix elements (i.e. if Condition (38) above holds,) the matricant series closes, and, as remarked above, the solution of the equation system becomes tantamount to finding the eigenvalues of the matrix A. This point will be discussed in further detail below.

The matricant method, although not necessarily practical in a given case, is nevertheless worth discussing because of the light it sheds on the nature of the solutions. Consider the general system of inhomogeneous equations in matrix form

$$\frac{du(z)}{dz} = A(z)\, u(z) + f(z) \tag{40}$$

where f(z) is a vector consisting of known "source" elements. Integrating (40) once, we have

$$u(z) = u(z_o) + \int_{z_o}^{z} dz'\, A(z')\, u(z') + \int_{z_o}^{z} f(z')\, dz' \quad . \tag{41}$$

Repeated substitution of u(z) as given by (41) into the integrand of the first integral on the right-hand side of (41) yields

$$u(z) = \Big[I + \int_{z_o}^{z} dz_1' \, A(z_1') + \int_{z_o}^{z} dz_1' \, A(z_1') \int_{z_o}^{z_1'} dz_2' \, A(z_2')$$

$$+ \int_{z_o}^{z} dz_1' \, A(z_1') \int_{z_o}^{z_1'} dz_2' \, A(z_2') \int_{z_o}^{z_2'} dz_3' \, A(z_3') + \ldots \Big] \, u(z_o)$$

$$+ \Big[\int_{z_o}^{z} f(z_1') \, dz_1' + \int_{z_o}^{z} dz_1' \, A(z_1') \int_{z_o}^{z_1'} dz_2' f(z_2')$$

$$+ \int_{z_o}^{z} dz_1' \, A(z_1') \int_{z_o}^{z_1'} dz_2' \, A(z_2') \int_{z_o}^{z_2'} dz_3' f(z_3') + \ldots \Big] \qquad (42)$$

where I is the identity matrix.

Let us find "Green's function" of the equation system (40) i.e. the response to forcing functions f(z) that are impulses at $z = z_1$, where $z_1 > z_o$. We write

$$f(z) = f_1 \, \delta(z - z_1) \; ; \; z_o < z_1 < z \qquad (43)$$

where f_1 is a constant, and we choose z_o at a point where

$$u(z_o) = 0 \qquad (44)$$

which is always allowable since the choice of z_o is perfectly arbitrary.

In this case, (42) has the form

$$u(z) = \Big[I + \int_{z_1}^{z} dz' \, A(z_1') + \int_{z_1}^{z} dz_1' \, A(z_1') \int_{z_1}^{z_1'} dz_2' \, A(z_2') + \ldots \Big] f_1 \quad . \qquad (45)$$

From (42) and (45), we see that solving the homogeneous equation (37) for a fixed set of values of the elements of $u(z_o)$ is equivalent to solving the inhomogeneous equation (40) with $u(z_o) = 0$ and with impulse sources at $z = z_1$, where $z_o < z_1 < z$. The method to be used here is that of assigning values to $u(z_o)$ and solving the homogeneous equation. Our solutions, then, will relate to a fixed set of values of $u(z_o)$ with the "forcing function" f(z) set equal to zero, and we will call z_o the "position of the source".

Referring back to (42) with $f(z) = 0$, it is not difficult to see that, if the condition (38) holds, then the third term of the matricant series is

$$\int_{z_o}^{z} dz_1' A(z_1') \int_{z_o}^{z_1'} dz_2' A(z_2') = \frac{1}{2!} \int_{z_o}^{z} dz_1' dz_2' A(z_1') A(z_2') =$$

$$\frac{1}{2!} \left(\int_{z_o}^{z} dz' A(z') \right)^2 \tag{46}$$

and the general term of the series is

$$\int_{z_o}^{z} dz_1' A(z_1') \int_{z_o}^{z_1'} dz_2' A(z_2') \cdots \int_{z_o}^{z_{N-1}'} dz_N A(z_N') = \frac{1}{N!} \left(\int_{z_o}^{z} dz' A(z') \right)^N . \tag{47}$$

The series then becomes the matrix analog of the power series for an exponential function and is denoted by

$$u(z) = \left(e^{\int_{z_o}^{z} A(z')dz'} \right) u(z_o) . \tag{48}$$

In the particular case where the matrix elements are constant between z_o and z,

$$u(z) = e^{A(z - z_o)} u(z_o) . \tag{49}$$

The exponential function of a matrix B can be evaluated from Sylvester's Theorem[12] obtainable by Laplace transform techniques[16] and has the form

$$e^{A(z - z_o)} = \sum_{k=1}^{n} \frac{e^{\lambda_k (z - z_o)}}{\prod_{\ell \neq k} (\lambda_\ell - \lambda_k)} \prod_{\ell \neq k} [\lambda_\ell I - A] \tag{50}$$

where the λ_k's are the n independent (non-degenerate) eigenvalues of the matrix A. If degenerate eigenvalues exist, then a modification of (50) is required, as discussed by Rekoff[16].

The solution of (37) with condition (38) in effect, then, is given by (50), with modifications if degenerate eigenvalues exist, and its determination

requires that the eigenvalues of B be found. If A is independent of z, the eigenvalues of B are those of $A(z-z_o)$; hence the problem reduces to that of finding the eigenvalues of A. This is the same problem that must be solved in a dispersion analysis, where the entire system of equations is complex-Fourier or Laplace transformed and is therefore reduced to a homogeneous set of simultaneous algebraic equations. The condition required for existence of nontrivial solutions of this system is the vanishing of the determinant of coefficients. This condition is exactly the same as the indicial polynomial equation, whose solution yields the eigenvalues of the A-matrix.

If the elements of the A-matrix have certain simple z-variations, then solutions can be obtained through integral transforms or orthogonal function expansions.

We have investigated a number of such possibilities and found them all wanting in one way or another. For example, a stratified medium theory can be formulated in which the z-variation of the elements of A(z) is linear or exponential. In the former case, Laplace transformation leads to a first order differential equation system with coefficients that are linear in s, the transform variable. Thus the problem is not reduced to algebra as it is in the constant coefficient case, but rather to a system of differential equations of the same degree of difficulty as the original system. In the case of exponential elements of A(z), the transformation leads to difference equations which can be solved numerically. Since inversion would be required after solution, in order to transform from the s-domain back to the z-domain, this did not seem like a satisfactory method. The same sort of limitations seemed to exist in the use of orthogonal function expansions. The case of

periodic coefficients, for example, can be treated by expanding both the elements of A(z) and the solutions in Fourier series on z, but the result is a set of coupled recursion relations for the coefficients, instead of the simultaneous equations that would appear if the A-elements were constant. These recursion relations could be solved but the numerical work required to obtain the Fourier coefficients of the A(z) elements would be considerable.

Another approach that was attempted in the early stages of investigation was reduction (by algebra and differentiation) of the original 20-equation system down to single differential or integro-differential equations of high order in single unknowns. This method is feasible and perfectly valid, but it has some very serious practical limitations if the coefficients have arbitrary z-variation. First, the coefficients obtained for the resulting differential equations are extremely complicated algebraically and the probability of eliminating errors in setting up computer solutions would be small. Secondly, in order to solve the high order differential equations (e.g. $12^{\underline{th}}$ order, as in one such development that was carried out), it might be necessary to reduce them to sets of coupled first order differential equations, which amounts to traveling in a circle.

Thirdly, the coefficients of the equations finally obtained contain many z-derivatives of the ambient gas parameters. This computation of these derivatives would be required before the coefficients of the differential equations could be specified. The derivatives, whose evaluation would require an enormous amount of computational labor, would be highly inaccurate since they would be obtained from very crude experimentally derived data curves. Thus a method that contains as few z-derivatives as possible in the co-

efficients is most desirable from the viewpoint of accuracy.

4.1.3. The Runge-Kutta Step-by-Step Method

No general solution obtainable by purely analytical methods was found that is suitable for the general problem, e.g. cases where the z-variation of the elements of A(z) is obtained from experimental data and where these elements are allowed to have any arbitrary z-variation and do not obey the condition (38). Such is the case in an accurate study of wave motions in the ionosphere; hence, we will invoke a purely numerical (step-by-step) technique which does not lead to convergence problems even if the elements have a particularly complicated variation with z.

After studying the relative advantages of various numerical techniques for solution of the equation system[17, 18] in the general case, it was decided to use the Runge-Kutta step-by-step procedure. This method has the advantage of high attainable accuracy without a prohibitively large expenditure of computer time.

To apply the Runge-Kutta method to (37), we first choose a z-interval h. We then begin with a set of values of the elements of the vector u(z) at $z = z_o$. The i^{th} element is denoted in general by $u_i(z)$. Its value at $z = z_o$ is denoted by $u_{i,o}$.

The long-hand expression for the matrix equation (37) is

$$\frac{du_i(z)}{dz} = \sum_{j=1}^{N} A_{ij}(z)\, u_j(z) \; ; \; i = 1, \ldots, N \quad . \tag{37}'$$

The Runge-Kutta formula for the value of $u_i(z)$ at $z = z_1 = z_o + h$, to be denoted by $u_{i,1}$, is[17]

$$u_{i,1} = u_{i,o} + \frac{1}{6}\left(k_{i,1}^{(1)} + 2k_{i,2}^{(1)} + 2k_{i,3}^{(1)} + k_{i,4}^{(1)}\right) \tag{51}$$

where

$$k_{i,1}^{(1)} = h \sum_{j=1}^{N} A_{ij}(z_o)u_{j,o}$$

$$k_{i,2}^{(1)} = h \sum_{j=1}^{N} A_{ij}(z_o + \frac{h}{2})\left[u_{j,o} + \frac{k_{j,1}^{(1)}}{2}\right]$$

$$k_{i,3}^{(1)} = h \sum_{j=1}^{N} A_{ij}(z_o + \frac{h}{2})\left[u_{j,o} + \frac{k_{j,2}^{(1)}}{2}\right]$$

$$k_{i,4}^{(1)} = h \sum_{j=1}^{N} A_{ij}(z_o + h)\left[u_{j,o} + k_{j,3}^{(1)}\right] .$$

The values of $u_i(z)$ at $z = z_o + (p + 1)h$, where p is a positive integer, are denoted by $u_{i,p+1}$ and are given by

$$u_{i,p+1} = u_{i,p} + \frac{1}{6}\left(k_{i,1}^{(p+1)} + 2k_{i,2}^{(p+1)} + 2k_{i,3}^{(p+1)} + k_{i,4}^{(p+1)}\right) \tag{51}'$$

where

$$k_{i,1}^{(p+1)} = h \sum_{j=1}^{N} A_{ij}(z_o + ph)u_{j,p}$$

$$k_{i,2}^{(p+1)} = h \sum_{j=1}^{N} A_{ij}(z_o + [p + \frac{1}{2}]h)\left(u_{j,p} + \frac{k_{j,1}^{(p+1)}}{2}\right)$$

$$k_{i,3}^{(p+1)} = h \sum_{j=1}^{N} A_{ij}(z_o + [p + \frac{1}{2}]h)\left(u_{j,p} + \frac{k_{j,2}^{(p+1)}}{2}\right)$$

$$k_{i,4}^{(p+1)} = h \sum_{j=1}^{N} A_{ij}(z_o + [p + 1]h)\left(u_{j,p} + k_{j,3}^{(p+1)}\right) .$$

Beginning with values of $u_{j,o}$ for $j = 1, \ldots, N$, we can proceed to find the successive values of $u_{j,1}, u_{j,2}, \ldots, u_{j,p+1}$ to any desired number of steps by the use of eq. (51)'. The accuracy attainable by this method is

fourth order in h. The sharper the gradients in the elements of A_{ij}, the smaller the value of h required for a given level of accuracy.

A Fortran program (soon to be run on the IBM-7090) has been prepared based on the coupling of a simultaneous equations routine with a Runge-Kutta routine. In effect, this program implements the steps between eqns. (35) and (37), then solves (37) by the Runge-Kutta routine. We originally carried out the steps from (36) to (37) algebraically, resulting in eqn. (37) where the elements of u(z) are the 10 variables $\hat{U}_{nz}$, $\hat{U}_{pz}$, $\hat{P}_{n1}$, $\hat{P}_{p1}$, $\hat{P}_{\ell 1}$, $\hat{E}_x$, $\hat{E}_y$, $\hat{H}_{1x}$, $\hat{J}_z$ and $\hat{H}_{1y}$. It was then decided that the direct computer solution of the entire system is more efficient, because some of the 11 variables that appear only algebraically ($\hat{U}_{nx}$, $\hat{U}_{ny}$, $\hat{U}_{px}$, $\hat{U}_{py}$, $\hat{\mathcal{R}}_{n1}$, $\hat{J}_x$, $\hat{J}_y$, $\hat{\mathcal{R}}_{c1}$, $\hat{E}_z$, $\hat{B}_{1z}$) will be desired as outputs, and the computer can produce them directly without difficulty.

By means of this program, it is possible to study many types of linear wave propagation in a neutral-electron-ion gas that is non-uniform in only one direction. Any desired sets of source functions, horizontal propagation constant $\hat{k}_x$, frequency ω and z-variation of ambient neutral and electron density, collision frequency and static magnetic field magnitude and direction can be studied by merely changing the input numbers. The computer outputs will consist of any desired components of the velocities $\hat{U}_n$ and $\hat{U}_p$, the current density $\hat{J}$, the perturbed electric and magnetic fields $\hat{E}$ and $\hat{B}_1$, the densities $\hat{\mathcal{R}}_{n1}$ or $\hat{\mathcal{R}}_{p1}$ or the pressures $\hat{P}_{n1}$, $\hat{P}_{p1}$, $\hat{P}_{\ell 1}$.

4.2. Applications

The possible applications of the analysis discussed in this report are manifold. The computer program is sufficiently general to cover many types

of linear wave propagation in a neutral electron-ion gas. By merely changing input parameter values, the computer can produce answers to such diverse questions as (a) the effect of acoustic-gravity waves on properties of the electron-ion plasma in a vertically inhomogeneous ionosphere, (b) the effects of vertical non-uniformity in electron density, collision frequency and/or ambient magnetic field on the propagation of MHD waves high in the ionosphere and that of electromagnetic waves in various parts of the ionosphere. In the case of electromagnetic waves, the point of view to be taken is that the neutral gas parameters $\hat{U}_{n1}$, $\hat{P}_{n1}$ and $\hat{\mathcal{R}}_{n1}$ are set to zero in the system equations* and the plasma equations are used to find a conductivity tensor. The latter is vertically non-uniform, its exact functional variation with z being determined by the z-variation of the static plasma parameters. The conductivity tensor is then substituted into the Maxwell equations IV and V in our system. This particular facet of vertically inhomogeneous ionospheric wave theory has been treated quite extensively in the literature. Examples of studies of radio wave propagation with altitude-varying conductivity tensor are provided by the work of Clemmow and Heading[19], Budden and Clemmow[20], Barron[21] and others.

The acoustic-gravity wave plasma interaction problem is based on an entirely different point-of-view. Over most of the ionosphere, the fractional ionization is so small that the terms in the neutral gas equations relating to the effects of the plasma are negligible, except at frequencies that are extremely low relative to neutral-plasma collision frequencies. The neutral gas wave

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* This reflects the fact that, because of low fractional ionization in the ionosphere, an electromagnetic wave would not significantly perturb the neutral gas.

propagation is thus only very weakly influenced by the presence of plasma. The neutral waves, whose equations are now uncoupled from the plasma equations, can by analyzed by the methods of Hines' 1960 paper[3] and the resulting neutral gas parameters $\hat{U}_{n1}$, $\hat{P}_{n1}$ and $\hat{\mathcal{R}}_{n1}$ can then be used as source terms in the plasma equations. In this way, the system, which in its general form is a system of 10 homogeneous differential equations and 11 homogeneous algebraic equations in a total of 21 unknowns, now becomes a system of 8 inhomogeneous differential equations and 8 algebraic equations in a total of 16 unknowns and with known source terms determined by solving the first 5 equations of the original system.

This technique, which takes advantage of the extremely low value of α that prevails in the ionosphere, reduces the magnitude of our computer problem in treating this particular ionospheric effect. A detailed discussion of the acoustic gravity wave plasma interaction problem will be presented in a later report.

Our computer program is by no means restricted to the ionosphere. It can be applied to any linear neutral-electron-ion gas with static parameters that are non-uniform in a single direction.* The normalization of the parameters in our equations would enhance the convenience of studying such gases, i.e., the normalized parameters could be varied in such a way as to produce universal curves. Wave problems that involve complicated one-dimensional spatial variations of parameters, simple degenerate cases of which have been solved by purely analytical methods, can be treated by our computer program without the necessity for either drastically simplifying the parameter variations or resorting to perturbation techniques. The latter, of course, are

* Provided, of course, that the assumptions and approximations used here are applicable.

severely limited in accuracy when the departure from the idealized case is large.

TABLE 1

Ordering of Equations and Vectors in Equations (34) and (35)

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_8 \\ u_9 \\ u_{10} \end{bmatrix} \quad \text{where} \quad \begin{aligned} u_1 &= \hat{P}_{n1} \\ u_2 &= \hat{P}_{p1} \\ u_3 &= \hat{P}_{\ell 1} \\ u_4 &= \hat{U}_{nz} \\ u_5 &= \hat{U}_{pz} \\ u_6 &= \hat{E}_y \\ u_7 &= \hat{E}_x \\ u_8 &= \hat{B}_{1y} \\ u_9 &= \hat{B}_{1x} \\ u_{10} &= \hat{J}_z \end{aligned}$$

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{10} \\ v_{11} \end{bmatrix} \quad \text{where} \quad \begin{aligned} v_1 &= \hat{U}_{nx} \\ v_2 &= \hat{U}_{ny} \\ v_3 &= \hat{U}_{px} \\ v_4 &= \hat{U}_{py} \\ v_5 &= \hat{R}_{n1} \\ v_6 &= \hat{R}_{p1} \\ v_7 &= \hat{J}_x \\ v_8 &= \hat{J}_y \\ v_9 &= \hat{R}_{c1} \\ v_{10} &= \hat{E}_z \\ v_{11} &= \hat{B}_{1z} \end{aligned}$$

Ordering of Equations

Equation 34

Eq. (1) $\hat{\hat{I}}$-n-z

(2) $\hat{\hat{I}}$-p-z

(3) $\hat{\hat{I}}$-0-z

(4) $\hat{\hat{II}}$-n

(5) $\hat{\hat{II}}$-p

(6) $I\hat{\hat{V}}$-x

(7) $I\hat{\hat{V}}$-y

(8) $\hat{\hat{V}}$-x

(9) $\hat{\hat{V}}$-y

(10) $\hat{\hat{II}}$-0

Equation 35

Eq. (1) $\hat{I}$-n-x

(2) $\hat{\hat{I}}$-n-y

(3) $\hat{I}$-p-x

(4) $\hat{\hat{I}}$-p-y

(5) $\hat{\hat{I}}$-0-x

(6) $\hat{\hat{I}}$-0-y

(7) $\hat{III}$-n

(8) $\hat{\hat{III}}$-p

(9) $\hat{\hat{III}}$-0

(10) $\hat{\hat{IV}}$-z

(11) $\hat{\hat{V}}$-z

TABLE 2

Matrices in Equations 34 and 35

$$A' = \begin{bmatrix} A'_{11} & 0 & 0 & A'_{14} & A'_{15} & 0 & 0 & 0 & 0 & A'_{1,10} \\ 0 & A'_{22} & 0 & A'_{25} & A'_{25} & 0 & 0 & 0 & 0 & A'_{2,10} \\ 0 & 0 & A'_{33} & A'_{34} & A'_{35} & 0 & 0 & 0 & 0 & A'_{3,10} \\ 0 & 0 & 0 & A'_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A'_{55} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A'_{66} & 0 & 0 & A'_{69} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A'_{77} & A'_{78} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A'_{87} & A'_{88} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A'_{96} & 0 & 0 & A'_{99} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -A'_{10,10} \end{bmatrix}$$

TABLE 2 (cont'd)

Matrices in Equations 34 and 35

$$B' = \begin{bmatrix} 0 & 0 & 0 & 0 & B'_{15} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B'_{26} & B'_{27} & B'_{28} & 0 & B'_{2,10} & 0 \\ 0 & 0 & B'_{33} & B'_{34} & 0 & 0 & B'_{37} & B'_{38} & B'_{39} & B'_{3,10} & 0 \\ B'_{41} & 0 & 0 & 0 & B'_{45} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B'_{53} & 0 & 0 & B'_{56} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B'_{7,10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B'_{87} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & B'_{98} & 0 & 0 & B'_{9,11} \\ 0 & 0 & 0 & 0 & 0 & 0 & B'_{10,7} & 0 & B'_{10,9} & 0 & 0 \end{bmatrix}$$

TABLE 2 (cont'd)

Matrices in Equations 34 and 35

$$C' = \begin{bmatrix} C'_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C'_{32} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C'_{3,10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C'_{4,10} \\ 0 & 0 & C'_{53} & 0 & C'_{55} & 0 & C'_{57} & 0 & 0 & C'_{5,10} \\ 0 & 0 & 0 & 0 & C'_{65} & C'_{66} & 0 & 0 & 0 & C'_{6,10} \\ C'_{71} & 0 & 0 & C'_{74} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C'_{82} & 0 & 0 & C'_{85} & 0 & 0 & 0 & 0 & C'_{8,10} \\ 0 & 0 & C_{93} & 0 & C'_{95} & 0 & 0 & 0 & 0 & C'_{9,10} \\ 0 & 0 & 0 & 0 & 0 & C'_{10,6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C'_{11,8} & 0 & C'_{11,10} \end{bmatrix}$$

TABLE 2 (cont'd)

Matrices in Equations 34 and 35

$$D' = \begin{bmatrix}
D'_{11} & 0 & D'_{13} & 0 & 0 & 0 & D'_{17} & 0 & 0 & 0 & 0 \\
0 & D'_{22} & 0 & D'_{24} & 0 & 0 & 0 & D'_{28} & 0 & 0 & 0 \\
D'_{31} & 0 & D'_{33} & 0 & 0 & 0 & D'_{37} & D'_{38} & 0 & 0 & 0 \\
0 & D'_{42} & 0 & D'_{44} & 0 & 0 & D'_{47} & D'_{48} & 0 & 0 & 0 \\
D'_{51} & 0 & D'_{53} & D'_{54} & 0 & 0 & D'_{57} & D'_{58} & 0 & 0 & 0 \\
0 & D'_{62} & D'_{63} & D'_{64} & 0 & 0 & D'_{67} & D'_{68} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & D'_{75} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & D'_{86} & 0 & 0 & D'_{89} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & D'_{96} & 0 & 0 & D'_{99} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D'_{10,11} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D'_{11,10} & 0
\end{bmatrix}$$

where:

$$A'_{11} = -(\hat{k}_{\rho n} + 2\hat{k}_{cn}), \quad A'_{22} = -(\hat{k}_{\rho p} + \hat{2}k_{cp}) = A'_{33}$$

$$A'_{14} = +\frac{i\gamma}{\hat{c}^2_{sno}}\left[1 + i\alpha(\hat{\nu}_{in} + \xi\hat{\nu}_{en})\right] = -D'_{11} = -D'_{22}$$

$$A'_{15} = +\frac{\gamma\alpha}{\hat{c}^2_{sno}}(\hat{\nu}_{in} + \xi\hat{\nu}_{en}) = -D'_{13} = -D'_{24}$$

$$A'_{1,10} = -\frac{\gamma\alpha\xi}{\hat{c}^2_{sno}}(\hat{\nu}_{in} - \hat{\nu}_{in}) = -D'_{17} = -D'_{28}$$

$$A'_{24} = +\frac{\gamma}{\hat{c}^2_{spo}}(\hat{\nu}_{in} + \xi\hat{\nu}_{en}) = -D'_{31} = -D'_{42}$$

$$A'_{25} = +\frac{i\gamma}{\hat{c}^2_{spo}}\left[1 + i(\hat{\nu}_{in} + \xi\hat{\nu}_{en})\right] = -D'_{33} = -D'_{44}$$

$$A'_{2,10} = -A'_{34} = \frac{\gamma\xi}{\hat{c}^2_{spo}}(\hat{\nu}_{en} - \hat{\nu}_{in}) = A'_{35} = -D'_{37} = -D'_{48} = D'_{51} = -D'_{53} = D'_{62} = -D'_{64}$$

$$A'_{3,10} = +\frac{i\gamma\xi}{\hat{c}^2_{spo}}\left[1 + i(\hat{\nu}_{en} + \xi\hat{\nu}_{in} + \hat{\nu}_{ei})\right] = -D'_{57} = -D'_{68}$$

$$A'_{44} = -\hat{k}_{\rho n}$$

$$A'_{55} = -\hat{k}_{\rho p} = A'_{66} = A'_{77} = A'_{88} = A'_{99} = A'_{10,10}$$

$$A'_{69} = -i = -A'_{78} = -A'_{87} = A'_{96} = -B'_{45} = -B'_{56} = -D'_{11,10} = -B'_{10,9}$$

$$B'_{15} = -\frac{\gamma\hat{g}}{\hat{c}^2_{sno}}$$

$$B'_{26} = -\frac{\gamma\hat{g}}{\hat{c}^2_{spo}}$$

$$B'_{27} = -\frac{i\gamma\xi\,\hat{\omega}_{ce}}{\hat{c}^2_{spo}}n_{By} = -iB'_{33} = iB'_{37} = -iC'_{55} = C'_{3,10} = iC'_{5,10}$$

$$B'_{28} = + \frac{i\gamma\xi\,\hat{\omega}_{ce}}{\hat{c}^2_{spo}}\, n_{Bx} = - i B'_{34} = i B'_{38} = - i C'_{65} = C'_{4,10} = i C'_{6,10}$$

$$B'_{39} = - \frac{\gamma \hat{g} \xi}{\hat{c}^2_{spo}}$$

$$B'_{3,10} = + \frac{\gamma\xi\,\hat{\omega}_p^{\,2}}{\hat{c}^2_{spo}} = - C'_{57} = - C'_{66}$$

$$B'_{41} = - i \hat{k}_x = B'_{53} = - B'_{7,10} = - B'_{9,11} = - C'_{11} = - C'_{32} = - C'_{53} = - i C'_{10,6}$$

$$= - C'_{11,8} = B'_{10,7}$$

$$B'_{87} = - 1 = - B'_{98} = - C'_{71} = - C'_{82} = D'_{10,11} = C'_{11,10}$$

$$C'_{74} = + i \left[\hat{k}_{\rho n} (1 - \gamma) + 2 \hat{k}_{cn} \right]$$

64 $$C'_{85} = + i \left[\hat{k}_{\rho p} (1 - \gamma) + 2 \hat{k}_{cp} \right] = - C'_{95} = - \left(\frac{1 + Z}{Z} \right) C'_{8,10} = C'_{9,10}$$

$$C'_{93} = \left(\frac{1 + Z}{Z} \right)$$

$$D'_{38} = \frac{i\gamma\xi\,\hat{\omega}_{ce}}{\hat{c}^2_{spo}}\, n_{Bz} = - D'_{47} = - i D'_{54} = i D'_{58} = i D'_{63} = - i D'_{67}$$

$$D'_{75} = - \gamma = D'_{86} = - D'_{96} = D'_{99}$$

$$D'_{89} = \frac{\gamma Z}{(1 + Z)}$$

REFERENCES

1. B.S. Tanenbaum and D. Mintzer, "Wave Propagation in a Partly Ionized Gas," Physics of Fluids, 5, 1226 (1962).

2. Final Report, ARPA Order No. 202-62, Amendment No. 4, Task No. 2, Project Code No. 2720, Applied Research Laboratory, Sylvania Electronic Systems, 26 June 1964, Sections 1 through 10 and Appendix A.

3. C.O. Hines, "Internal Atmospheric Gravity Waves at Ionospheric Heights," Can. J. Physics, Vol. 38 (1960).

4. M.L.V. Pitteway and C.O. Hines, "The Viscous Damping of Atmospheric Gravity Waves," Can. J. Physics, Vol. 41, (1963).

5. C.O. Hines, "Atmospheric Gravity Waves High in the Ionosphere," presented at the 46th Annual Meeting of the AGU, Washington, D.C., April 19-22, 1965.

6. L. Spitzer, "Physics of Fully Ionized Gases," Interscience, 1962, Appendix, P. 155.

7. T.G. Cowling, "Magnetohydrodynamics," Interscience, 1957, Section 6.4, P. 105-108.

8. E.A. Coddington and N. Levinson, "Theory of Ordinary Differential Equations," McGraw-Hill, 1955, Chapters 1, 2, 3.

9. F.R. Moulton, "Differential Equations," Dover, 1930, Chapters 14, 15, 17.

10. W. Kaplan, "Operational Methods for Linear Systems," Addison-Wesley, 1962, Chapter 8.

11. F.R. Gantmacher, "Matrix Theory," Chelsea, 1959, Vol. II, Chapter XIV.

12. A. Michal, "Matrix and Tensor Calculus," Wiley, 1947, Chapter 4.

13. J.T. Tou, "Modern Control Theory," McGraw-Hill, 1964, Chapter 3.

14. B.V. Kinariwala, "Analysis of Time-Varying Networks," IRE International Convention Record, 1961, P. 268.

15. L. Zadeh and C. Desoer. "Linear System Theory," State-Space Approach, McGraw-Hill, 1963.

16. M. Rekoff, "State Variable Techniques for Control Systems," Electro-Technology, May 1964, P. 36-47.

17. A. Bennett, W. Milne, H. Bateman, "Numerical Integration of Differential Equations," Dover, New York, 1956.

18. L.D. Harris, "Numerical Methods Using Fortran," C. Merrill Books, Inc., Columbus, Ohio, 1964.

19. P.C. Clemmow and J. Heading, "Coupled Form of the Differential Equations Governing Radio Propagation in the Ionosphere," Proc. Camb. Phil. Soc., 50 (1954), 319.

20. K. G. Budden and P.C. Clemmow, "Coupled Form of the Differential Equations Governing Radio Propagation in the Ionosphere II," Proc. Camb. Phil. Soc., 50 (1956).

21. D W. Barron, "The Numerical Solution of Differential Equations Governing the Reflection of Long Radio Waves from the Ionosphere IV," Proc. Royal Soc., 1960.

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13 ABSTRACT

The problem of wave propagation in a linearized 3-component (electron, ion, neutral) gas is discussed for the case where the ambient gas parameters are arbitrary functions of the vertical coordinate in z and uniform with respect to time and the horizontal coordinates. The formulation of the problem from the Boltzmann equation to a system of simultaneous coupled linear algebraic and first order differential equations in z is discussed in some detail in the first part of the report. In the second part, mathematical methods of dealing with such systems are treated, culminating in a discussion of the numerical method finally chosen and the computer program used to obtain solutions.

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